

Fibonacci Day maths for IB Maths SL and HL: answers

IB Math Revision · IB Maths SL and HL (AA and AI) · 23 November · Teacher answers and notes

The starter is non-calculator. A GDC helps in tasks A and B.

Teacher notes

- Fibonacci Day is 23 November because 11/23, written month first, reads 1, 1, 2, 3.
- In task A (b), insist on the reason for rejecting $(1 - \sqrt{5})/2$: every ratio is positive.
- Task B (d) makes a good display: students draw the squares on squared paper and add the arcs with compasses.
- Extension 1 explains task A: $|\psi| < 1$, so $\psi^n \rightarrow 0$ and $F_{n+1}/F_n \rightarrow \phi$.

Starter

1. 2, 3, 5, 8, 13, 21, 34, 55

- Add the two previous terms each time: $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, and so on up to $21 + 34 = 55$.

2. $x = (1 + \sqrt{5})/2$ or $(1 - \sqrt{5})/2$

- $x^2 - x - 1 = 0$, so $x = (1 \pm \sqrt{(1 + 4)})/2 = (1 \pm \sqrt{5})/2$.

3. 20, which is $F_8 - 1$

- $1 + 1 + 2 + 3 + 5 + 8 = 20$
- $F_8 = 21$, so the sum is one less than F_8 .

4. 1.615

- $21 \div 13 = 1.6153\dots$

Task A: The limit of the ratios

a. 1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, 1.619: the ratios settle down, alternately above and below about 1.618

- $1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, 34/21$
- $r_8 = 34 \div 21 = 1.619$ (3 d.p.)

b. $\phi = (1 + \sqrt{5})/2 \approx 1.618$

- $F_{n+2}/F_{n+1} = 1 + F_n/F_{n+1}$, that is $r_{n+1} = 1 + 1/r_n$.
- In the limit $\phi = 1 + 1/\phi$, so $\phi^2 = \phi + 1$ and $\phi = (1 \pm \sqrt{5})/2$.
- Every ratio is positive, so $\phi = (1 + \sqrt{5})/2 = 1.618$ (3 d.p.).

c. $\phi^3 = 2\phi + 1$, $\phi^4 = 3\phi + 2$: Fibonacci numbers again

- $\phi^3 = \phi \times \phi^2 = \phi^2 + \phi = (\phi + 1) + \phi = 2\phi + 1$.
- $\phi^4 = \phi \times \phi^3 = 2\phi^2 + \phi = 2(\phi + 1) + \phi = 3\phi + 2$.
- In general $\phi^n = F_n\phi + F_{n-1}$.

Task B: Golden rectangles and the spiral

a. **$1 \div (\varphi - 1) = \varphi$, because $\varphi(\varphi - 1) = 1$**

- $\varphi(\varphi - 1) = \varphi^2 - \varphi = (\varphi + 1) - \varphi = 1$.
- So $1 \div (\varphi - 1) = \varphi$: the long side is φ times the short side.

b. **48.5 cm**

- $30\varphi = 30 \times 1.6180... = 48.54...$
- 48.5 cm (3 s.f.)

c. **$273 \text{ cm}^2 = 13 \times 21$**

- $1 + 1 + 4 + 9 + 25 + 64 + 169 = 273$
- $13 \times 21 = 273$. In general $F_1^2 + ... + F_n^2 = F_n F_{n+1}$.

d. **$33\pi/2 \approx 51.8 \text{ cm}$**

- Each quarter circle has length $\frac{1}{4} \times 2\pi r = \pi r/2$.
- Total = $\pi/2 \times (1 + 1 + 2 + 3 + 5 + 8 + 13) = \pi/2 \times 33 = 33\pi/2 = 51.83...$

Task C: Fibonacci identities

a. **$143 = 144 - 1$**

- $1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 = 143$
- $F_{11} = 89$ and $F_{12} = 144$, so $F_{12} - 1 = 143$.

b. **$1, -1, 1, -1, 1$: the result is $(-1)^n$**

- $n = 2$: $1 \times 2 - 1 = 1$. $n = 3$: $1 \times 3 - 4 = -1$. $n = 4$: $2 \times 5 - 9 = 1$.
- $n = 5$: $3 \times 8 - 25 = -1$. $n = 6$: $5 \times 13 - 64 = 1$.
- So $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

c. **10**

- The pattern of odd (O) and even (E) is O, O, E, O, O, E, ..., because $O + O = E$, $O + E = O$ and $E + O = O$.
- So F_n is even exactly when n is a multiple of 3: $n = 3, 6, ..., 30$, which is 10 numbers.

Extension

1. **$F_{20} = 6765$**

- $n = 1$: $(\varphi - \psi)/\sqrt{5} = \sqrt{5}/\sqrt{5} = 1$.
- $n = 2$: $(\varphi^2 - \psi^2)/\sqrt{5} = (\varphi - \psi)(\varphi + \psi)/\sqrt{5} = \sqrt{5} \times 1/\sqrt{5} = 1$.
- $n = 20$: $(\varphi^{20} - \psi^{20})/\sqrt{5} = 6765$.

2. **Proof**

- Base case: $F_1 = 1 = F_3 - 1$.
- Assume the sum of the first k terms is $F_{k+2} - 1$. Adding F_{k+1} gives $F_{k+2} + F_{k+1} - 1 = F_{k+3} - 1$.
- True for $n = 1$, and true for $n = k + 1$ whenever it is true for $n = k$, so true for every positive integer n .

Every question is original and every answer was recomputed by computer before publishing.