

Number and algebra

IB Maths AA HL · Knowledge organiser

KEY DEFINITIONS

Complex number

$z = a + bi$, where $i^2 = -1$; a is the real part and b the imaginary part.

Modulus and argument

$|z|$ is the distance of z from the origin on an Argand diagram; $\arg z$ is the angle it makes with the positive real axis.

Proof by induction

Show a statement is true for the first case, then that true for $n = k$ means true for $n = k + 1$.

Permutation

An arrangement where order matters: ${}^nP_r = n!/(n-r)!$.

YOU SHOULD BE ABLE TO...

- Recognise an arithmetic sequence from its common difference and use $u_n = u_1 + (n-1)d$ to find a term, the first term or the number of terms.
- Simplify with the laws of exponents, including negative and fractional powers, and solve exponential equations by matching bases or taking logarithms.
- Expand expressions such as $(1+x)^{-1}$ and $(4-x)^{1/2}$ as infinite series, stating the values of x for which each expansion is valid.
- Plot complex numbers on an Argand diagram, find the modulus and argument, and convert between Cartesian and $r(\cos \theta + i \sin \theta)$ forms.
- Count arrangements with restrictions, such as items together, apart or in fixed positions, using nP_r and nC_r .
- Prove results such as the irrationality of $\sqrt{2}$ by contradiction, and disprove false statements by finding a single counterexample.

COMMON MISTAKES

- ✗ Proof by induction: weak structure and "let $n = k$ "
- ✗ Complex numbers and polynomials: conjugate roots, all the roots, general solutions
- ✗ Confusing common difference d with common ratio r
- ✗ Using the arithmetic sum formula for a geometric series

KEY FORMULAS

B = given in the exam (formula booklet or list); learn the rest.

Arithmetic **B** $u_n = u_1 + (n-1)d$,
 $S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$

Geometric **B** $u_n = u_1 r^{n-1}$, $S_n = \frac{u_1(1-r^n)}{1-r}$, $S_\infty = \frac{u_1}{1-r}$ ($|r| < 1$)

Compound interest **B** $FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$

Exponents & logs **B** $a^x = b \iff x = \log_a b$, $a^x = e^{x \ln a}$,
 $\log_a a^x = x = a^{\log_a x}$

Log laws **B** $\log_a xy = \log_a x + \log_a y$, $\log_a \frac{x}{y} = \log_a x - \log_a y$,
 $\log_a x^m = m \log_a x$

Change of base **B** $\log_b a = \frac{\log_c a}{\log_c b}$

Binomial, $n \in \mathbb{N}$ **B** $(a+b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Permutations **B** ${}^nP_r = \frac{n!}{(n-r)!}$

Binomial, $n \in \mathbb{Q}$, $|x| < 1$ **B** $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$

$(a+bx)^n = a^n\left(1 + \frac{b}{a}x\right)^n$, valid for $|x| < \left|\frac{a}{b}\right|$

Partial fractions $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$

Complex numbers **B** $z = a + bi = r(\cos \theta + i \sin \theta) = re^{i\theta} = r \operatorname{cis} \theta$

Modulus, argument, conjugate $r = |z| = \sqrt{a^2 + b^2}$, $\tan \theta = \frac{b}{a}$, $z^* = a - bi$,
 $zz^* = |z|^2$

Products $|z_1 z_2| = |z_1| |z_2|$, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$,
 $\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2$

De Moivre **B** $[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta) = r^n e^{in\theta}$

n th roots of $re^{i\theta}$ $r^{1/n} e^{i(\theta+2\pi k)/n}$, $k = 0, 1, \dots, n-1$

More on the full formula sheet.

WORKED EXAMPLE

Write $z = 1 + i\sqrt{3}$ in the form $re^{i\theta}$ and hence find z^3 .

1. $r = \sqrt{1^2 + (\sqrt{3})^2} = 2$ and $\theta = \arctan(\sqrt{3}/1) = \pi/3$

2. $z = 2e^{i(\pi/3)}$, so $z^3 = 2^3 e^{i(\pi)} = 8 \times (-1)$

Answer: $z = 2e^{i(\pi/3)}$ and $z^3 = -8$