Graphical Behaviour and The Second Derivative: exam practice

10 questions · 52 marks · worked solutions and markschemes

Name _____ Date _____ Score / 52

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/slaa/graphical-behaviour-and-the-second-derivative-exam-practice

1. **4 marks · Paper 1 style · no calculator**

For the function $f(x) = x^3 - 3x^2 + 5$, find the exact x-coordinate of the point of inflection.

2. **5 marks · Paper 1 style · no calculator**

Find the exact x-coordinates of the points of inflexion on the curve $y = x^2 e^{-x}$.

3. **4 marks · Paper 2 style · calculator**

Consider the function $f(x) = x^3 - 3x^2 - 9x + 2$.

1. Graph the function on your graphic display calculator and find the exact coordinates of the local maximum point.
2. Find the exact coordinates of the local minimum point.

4. **5 marks · Paper 2 style · calculator**

Let $h(x) = \sin x + \cos x$ for $0 \leq x \leq 2\pi$.

1. Find $h''(x)$.
2. A point of inflexion occurs where the second derivative is zero and changes sign (a change in concavity). Use your GDC to solve $h''(x) = 0$ and find the exact x-coordinates of the points of inflexion.

5. **5 marks · Paper 2 style · calculator**

A function f is defined for $x > 0$. The derivative of f is given by:

$$f'(x) = 3x^2 + \frac{2}{x^3} - 25$$

1. Find $f''(x)$.
2. The graph of f is concave up when $x > n$, where n is the least possible real number that makes this inequality true. Find the exact value of n.

6. **5 marks · Paper 2 style · calculator**

Let $f(x) = \dfrac{x}{x^2+1}$.

(a) Find the coordinates of the maximum point of the graph of f. **[2]**

(b) Use your calculator to find the x-coordinates of the three points of inflexion. **[3]**

7. **6 marks · Paper 2 style · calculator**

Let $f(x) = \ln(x^2+1)$.

(a) Show that $f''(x) = \dfrac{2(1-x^2)}{(x^2+1)^2}$. **[3]**

(b) Find the interval in which the graph of f is concave up. **[2]**

(c) Write down the x-coordinates of the points of inflexion. **[1]**

8. **7 marks · Paper 1 style · no calculator**

Let $f(x) = x^3 - 6x^2 + 9x + 1$.

(a) Find the coordinates of the stationary points and determine their nature. **[5]**

(b) Find the coordinates of the point of inflexion. **[2]**

9. **5 marks · Paper 1 style · no calculator**

The derivative of a function g is $g'(x) = (x - 1)(x + 2)^2$.

(a) Write down the x-coordinates of the stationary points of g. **[1]**

(b) Determine the nature of each stationary point. Justify your answers. **[4]**

10. **6 marks · Paper 1 style · no calculator**

Let $f(x) = x^4 - 4x^3$.

(a) Find $f''(x)$. **[2]**

(b) Find the coordinates of the points of inflexion, justifying your answer. **[4]**

Answers and worked solutions

Question 1

4 marks

Answer: $x = 1$

First derivative: $f'(x) = 3x^2 - 6x$ (A1)
Second derivative: $f''(x) = 6x - 6$ A1
Set the second derivative to zero: $6x - 6 = 0$ M1
$x = 1$ (f'' changes sign from negative to positive here, so it is a point of inflection) A1

Question 2

5 marks

Answer: $x = 2 + \sqrt{2}, x = 2 - \sqrt{2}$

Product rule:
$\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = e^{-x}(2x - x^2)$ M1 A1
Differentiate again:
$\frac{d^2y}{dx^2} = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) = e^{-x}(x^2 - 4x + 2)$
A1
Set $\frac{d^2y}{dx^2} = 0$: since $e^{-x} > 0$, $x^2 - 4x + 2 = 0$
M1
$x = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$ (distinct roots, so y'' changes sign at each) A1

Question 3

4 marks

Answer: (a) $(-1, 7)$; (b) $(3, -25)$

1. **CG50:** Go to `MENU 5 (Graph)`. Enter $Y1 = x^3 - 3x^2 - 9x + 2$. `DRAW`. (M1)
   Press `SHIFT -> F5 (G-Solv) -> F2 (MAX)`.
   The calculator displays $X = -1, Y = 7$.
   Coordinates: $(-1, 7)$. A1
2. Press `SHIFT -> F5 (G-Solv) -> F3 (MIN)`. (M1)
   The calculator displays $X = 3, Y = -25$.
   Coordinates: $(3, -25)$. A1

Question 4

5 marks

Answer: (a) $h''(x) = -sin(x) - cos(x)$; (b) $x = 3\pi/4, x = 7\pi/4$

1. $h'(x) = \cos x - \sin x$. (A1)
   $h''(x) = -\sin x - \cos x$. A1
2. Set
   $h''(x) = 0 \implies -\sin x - \cos x = 0 \implies \sin x = -\cos x \implies \tan x = -$
   . (M1)
   CG50: Use `MENU 5 (Graph)` or `Equation Solver` to solve $\tan x = -1$ for $0 \le x \le 2\pi$.
   Exact coordinates are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.
   A1A1

Question 5

5 marks

Answer: (a) $f''(x) = 6x - 6/x^4$; (b) $n = 1$

1. Rewrite $f'(x) = 3x^2 + 2x^{-3} - 25$. (M1)
   $f''(x) = 6x - 6x^{-4} = 6x - \frac{6}{x^4}$. A1
2. Concave up means $f''(x) > 0$. (M1)
   $6x - \frac{6}{x^4} > 0 \implies 6x > \frac{6}{x^4}$.
   Since $x > 0$, multiply by x^4:
   $6x^5 > 6 \implies x^5 > 1 \implies x > 1$. (M1)
   Therefore, the least value is $n = 1$. A1

Question 6

5 marks

Answer: (a) $(1,\ 0.5)$; (b) $x = -1.73,\ 0,\ 1.73$

(a)

$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2} = 0$ or GDC (M1)

$(1,\ 0.5)$ A1

(b)

$x = -1.73$ A1

$x = 0$ A1

$x = 1.73$ A1

Question 7

6 marks

Answer: (b) $-1 < x < 1$; (c) $x = -1$ and $x = 1$

(a)

$f'(x) = \frac{2x}{x^2 + 1}$ A1

quotient rule: $\frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2}$ M1

$= \frac{2 - 2x^2}{(x^2 + 1)^2}$ A1 AG

(b)

$1 - x^2 > 0$ **(M1)**

$-1 < x < 1$ **A1**

(c)

$x = \pm 1$ **A1**

Question 8 — 7 marks

Answer: (a) maximum $(1, 5)$, minimum $(3, 1)$; (b) $(2, 3)$

(a)

$f'(x) = 3x^2 - 12x + 9$ **A1**

$3(x - 1)(x - 3) = 0$ **M1**

$(1, 5)$ and $(3, 1)$ **A1**

$f''(x) = 6x - 12$; $f''(1) < 0$ so $(1, 5)$ is a maximum **R1**

$f''(3) > 0$ so $(3, 1)$ is a minimum **R1**

(b)

$6x - 12 = 0$ **(M1)**

$(2, 3)$ **A1**

Question 9 — 5 marks

Answer: (a) $x = 1$ and $x = -2$

(a)

$x = 1$ and $x = -2$ **A1**

(b)

g' changes from negative to positive at $x = 1$ **R1**

so $x = 1$ is a minimum **A1**

g' is negative on both sides of $x = -2$ **R1**

so $x = -2$ is a stationary point of inflexion **A1**

Question 10 — 6 marks

Answer: (a) $12x^2 - 24x$; (b) $(0, 0)$ and $(2, -16)$

(a)

$f'(x) = 4x^3 - 12x^2$ **A1**

$f''(x) = 12x^2 - 24x$ **A1**

(b)

$12x(x - 2) = 0$ **M1**

f'' changes sign at $x = 0$ and at $x = 2$ **R1**

$(0, 0)$ **A1**

$(2, -16)$ **A1**

Every solution is also on ibmathrevision.com/slaa/graphical-behaviour-and-the-second-derivative-exam-practice
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