Applications of Differentiation: exam practice

10 questions · 59 marks · worked solutions and markschemes

Name _____ Date _____ Score / 59

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/slaa/applications-of-differentiation-exam-practice

1. **4 marks · Paper 2 style · calculator**

The population of birds in a wildlife reserve is modelled by the function $P(t) = \frac{800}{1+15e^{-0.3t}}$ for $0 \leq t \leq 20$, where t is the time in months. Find the value of t for which the population of birds is increasing most rapidly.

2. **6 marks · Paper 1 style · no calculator**

A closed cylindrical tin is manufactured to have a fixed volume of $54\pi \text{ cm}^3$. Let its radius be $r \text{ cm}$ and its height be $h \text{ cm}$. Find the exact radius of the tin that minimises its total surface area.

3. **6 marks · Paper 1 style · no calculator**

Find the exact values of m for which the lines $y = mx - 2$ are tangents to the quadratic curve with equation $y = x^2 - 4x + 2$.

4. **4 marks · Paper 1 style · no calculator**

The profit, €P, from selling x items is $P(x) = -0.5x^2 + 40x - 300$.

(a) Find $P'(x)$. **[1]**

(b) Find the number of items that gives the greatest profit, and the greatest profit. **[3]**

5. **6 marks · Paper 2 style · calculator**

A farmer fences a rectangular field of area $800\ \text{m}^2$ next to a straight wall. The wall forms one side, and fencing is used for the other three sides. The two sides at right angles to the wall each have length x m.

(a) Show that the length of fencing needed is $L = 2x + \dfrac{800}{x}$. **[2]**

(b) Find the minimum length of fencing. **[4]**

6. **7 marks · Paper 1 style · no calculator**

An open box is made from a square sheet of card 12 cm by 12 cm by cutting a square of side x cm from each corner and folding up the sides.

(a) Show that the volume of the box is $V = 4x^3 - 48x^2 + 144x$. **[2]**

(b) Find the value of x that gives the maximum volume, and find this volume. **[5]**

7. **8 marks · Paper 1 style · no calculator**

Let $f(x) = x^3 - 2x + 1$.

(a) Find the equation of the tangent to the graph of f at the point where $x = 1$. **[3]**

(b) Find the equation of the normal at the same point. **[2]**

(c) The normal meets the graph of f again at two more points. Find their coordinates. **[3]**

8. **5 marks · Paper 1 style · no calculator**

The equation of a curve is $y = x^3 - 4x^2 + kx$, where k is a constant. The equation of the tangent to the curve at the point where $x = 2$ is $y = -3x$.

1. State the gradient of the tangent line.
2. Find $\frac{dy}{dx}$ in terms of x and k.
3. By substituting $x = 2$ into your derivative, find the exact value of k.

9. **6 marks · Paper 1 style · no calculator**

The point $P(x, \sqrt{x})$ lies on the curve $y = \sqrt{x}$. The point A has coordinates $(4, 0)$.

(a) Show that $AP^2 = x^2 - 7x + 16$. **[2]**

(b) Find the exact minimum distance from A to the curve. **[4]**

10. **7 marks · Paper 1 style · no calculator**

Let $f(x) = xe^{2x}$.

(a) Find $f'(x)$. **[2]**

(b) Find the exact coordinates of the stationary point. **[3]**

(c) Use the second derivative to show that this point is a minimum. **[2]**

Answers and worked solutions

Question 1

4 marks

Answer: $t = 9.03$

The population increases most rapidly when $P'(t)$ is a maximum, i.e. at the point of inflection of P (where $P''(t) = 0$) **M1**

Use the GDC: graph $P'(t)$ and find its maximum (or find the inflection point of $P(t)$) **M1**

$t = 9.03$ months (3 s.f.) $\left(= \frac{\ln 15}{0.3}\right)$ **A2**

Question 2

6 marks

Answer: $r = 3$ cm

Use the volume to find h:

$\pi r^2 h = 54\pi \implies h = \frac{54}{r^2}$ **M1**

Substitute into $S = 2\pi r^2 + 2\pi r h$:

$S(r) = 2\pi r^2 + \frac{108\pi}{r}$ **A1**

Differentiate: $S'(r) = 4\pi r - \frac{108\pi}{r^2}$ **A1**

Set $S'(r) = 0$: $4\pi r^3 = 108\pi$ **M1**

$r^3 = 27 \implies r = 3$ cm **A1**

$S''(r) = 4\pi + \frac{216\pi}{r^3} > 0$, so this is a minimum **R1**

Question 3

6 marks

Answer: $m = 0, m = -8$

Equate line and curve: $x^2 - 4x + 2 = mx - 2$ **M1**

Rearrange: $x^2 - (4+m)x + 4 = 0$ **A1**

Tangent means one repeated root, so the discriminant is zero **R1**

$(4+m)^2 - 4(1)(4) = 0$ **M1**

$(4+m)^2 = 16 \implies 4 + m = \pm 4$ **(A1)**

$m = 0$ or $m = -8$ **A1**

Question 4

4 marks

Answer: (a) $40 - x$; (b) 40 items, €500

(a)

$-x + 40$ **A1**

(b)

$-x + 40 = 0$ **M1**

$x = 40$ **A1**

€500 **A1**

Question 5

6 marks

Answer: (b) $80\ m$

(a)

side parallel to the wall $= \frac{800}{x}$ **M1**

$L = 2x + \frac{800}{x}$ **A1 AG**

(b)

$\frac{dL}{dx} = 2 - \frac{800}{x^2}$ **A1**

$2 - \frac{800}{x^2} = 0$ **M1**

$x = 20$ **A1**

$L = 80\ m$ **A1**

Question 6

7 marks

Answer: (b) $x = 2$, $V = 128\ \text{cm}^3$

(a)

$V = x(12 - 2x)^2$ **M1**

$= x(144 - 48x + 4x^2) = 4x^3 - 48x^2 + 144x$ **A1 AG**

(b)

$\frac{dV}{dx} = 12x^2 - 96x + 144$ **A1**

$12(x-2)(x-6) = 0$ **M1**

$x = 6$ gives $V = 0$, so $x = 2$ **R1**

$x = 2$ **A1**

$V = 128\ \text{cm}^3$ **A1**

Question 7

8 marks

Answer: (a) $y = x - 1$; (b) $y = -x + 1$; (c) $(0, 1)$ and $(-1, 2)$

(a)

$f'(x) = 3x^2 - 2$, $f'(1) = 1$ **A1**

$f(1) = 0$ **A1**

$y = x - 1$ **A1**

(b)

gradient -1 **(M1)**

$y = -x + 1$ **A1**

(c)

$x^3 - 2x + 1 = -x + 1$ **M1**

$x^3 - x = 0$, $x = 0,\ -1$ **A1**

$(0, 1)$ and $(-1, 2)$ **A1**

Question 8

5 marks

Answer: (a) -3; (b) $dy/dx = 3x^2 - 8x + k$; (c) $k = 1$

1. From $y = -3x$, the gradient is -3 **A1**
2. $\frac{dy}{dx} = 3x^2 - 8x + k$ **A1A1**
3. Gradient is -3 when $x = 2$:
   $3(2)^2 - 8(2) + k = -3$ **(M1)**
   $12 - 16 + k = -3 \Rightarrow k = 1$ **A1**
   (Check: when $x = 2$, $y = 8 - 16 + 2 = -6 = -3(2)$, so the point lies on the tangent.)

Question 9

6 marks

Answer: (b) $\frac{\sqrt{15}}{2}$

(a)

$AP^2 = (x - 4)^2 + (\sqrt{x})^2$ **M1**

$= x^2 - 8x + 16 + x = x^2 - 7x + 16$ **A1 AG**

(b)

$\frac{d}{dx}(AP^2) = 2x - 7 = 0$ **M1**

$x = \frac{7}{2}$ **A1**

$AP^2 = \frac{15}{4}$ **A1**

$AP = \frac{\sqrt{15}}{2}$ **A1**

Question 10

7 marks

Answer: (a) $(1 + 2x)e^{2x}$; (b) $\left(-\frac{1}{2},\ -\frac{1}{2e}\right)$

(a)

product rule **(M1)**

$f'(x) = e^{2x} + 2xe^{2x} = (1 + 2x)e^{2x}$ **A1**

(b)

$1 + 2x = 0$ since $e^{2x} > 0$ **M1**

$x = -\frac{1}{2}$ **A1**

$\left(-\frac{1}{2},\ -\frac{1}{2e}\right)$ **A1**

(c)

$f''(x) = (4 + 4x)e^{2x}$ **A1**

$f''\left(-\frac{1}{2}\right) = 2e^{-1} > 0$, so minimum **R1**

Every solution is also on ibmathrevision.com/slaa/applications-of-differentiation-exam-practice
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