Phase Portraits: exam practice

10 questions · 31 marks · worked solutions and markschemes

Name _____ Date _____ Score / 31

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlai/phase-portraits-exam-practice

1. **2 marks · Paper 2 style · calculator allowed**

A phase portrait shows that $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$. For $\dot{x} = x + 2y$, $\dot{y} = 2x + y$:

Find $\frac{dy}{dx}$ at $(1, 0)$ and at $(0, 1)$. **[2]**

2. **3 marks · Paper 2 style · calculator allowed**

A system is $\dot{x} = -x + 2y$, $\dot{y} = -2x - y$.

Find the eigenvalues and describe the phase portrait. **[3]**

3. **3 marks · Paper 2 style · calculator allowed**

A system is $\dot{x} = 2x$, $\dot{y} = -3y$.

Classify the equilibrium and describe the trajectories near the axes. **[3]**

4. **3 marks · Paper 2 style · calculator allowed**

A system is $\dot{x} = y$, $\dot{y} = -9x$.

Find the eigenvalues and describe the trajectories. **[3]**

5. **3 marks · Paper 2 style · calculator allowed**

For the system $\dot{x} = x - y$, $\dot{y} = x + y$.

Find the eigenvalues and classify the equilibrium. **[3]**

6. **3 marks · Paper 2 style · calculator allowed**

For the system $\dot{x} = -2x$, $\dot{y} = -5y$, a trajectory starts at $(1, 1)$.

Write down $x(t)$ and $y(t)$ and describe what happens as $t \to \infty$. **[3]**

7. **4 marks · Paper 2 style · calculator allowed**

A system is $\dot{x} = 3x + y$, $\dot{y} = x + 3y$.

Find the eigenvalues and eigenvectors, and classify the equilibrium. **[4]**

8. **4 marks · Paper 2 style · calculator allowed**

The system $\dot{\mathbf{x}} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\mathbf{x}$ has eigenvalues 3 and -1.

Find the eigenvectors and write the general solution. **[4]**

9. **3 marks · Paper 2 style · calculator allowed**

Two species have populations satisfying $\dot{x} = 0.4x - 0.2y$, $\dot{y} = 0.1x + 0.1y$ (deviations from equilibrium).

Find the eigenvalues and classify the equilibrium. **[3]**

10. **3 marks · Paper 2 style · calculator allowed**

Use Euler's method with $h = 0.1$ for $\dot{x} = y$, $\dot{y} = -x$, starting at $(1, 0)$, to estimate the position after two steps.

Find the estimate. **[3]**

Answers and worked solutions

Question 1 — 2 marks

Answer: 2; $\frac{1}{2}$

$\frac{2}{1} = 2$ **A1**

$\frac{1}{2}$ **A1**

Question 2 — 3 marks

Answer: $-1 \pm 2i$; stable spiral

$\lambda^2 + 2\lambda + 5 = 0$ **M1**

$\lambda = -1 \pm 2i$ **A1**

stable spiral, clockwise **A1**

Question 3 — 3 marks

Answer: a saddle point

$\lambda = 2,\ -3$ **A1**

saddle point **A1**

move away along the $x-$axis, towards the origin along the $y-$axis **A1**

Question 4 — 3 marks

Answer: $\pm 3i$; a centre

$\lambda^2 + 9 = 0$ **M1**

$\lambda = \pm 3i$ **A1**

centre: closed ellipses **A1**

Question 5 — 3 marks

Answer: $1 \pm i$; unstable spiral

$(1-\lambda)^2 + 1 = 0$ **M1**

$\lambda = 1 \pm i$ **A1**

unstable spiral **A1**

Question 6 — 3 marks

Answer: $x = e^{-2t}$, $y = e^{-5t}$; to the origin

$x = e^{-2t}$ **A1**

$y = e^{-5t}$ **A1**

both tend to 0: stable node **A1**

Question 7 — 4 marks

Answer: 4, 2; unstable node

$\lambda = 4$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **A1A1**

$\lambda = 2$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ **A1**

unstable node **A1**

Question 8 — 4 marks

Answer: a proof: the worked solution above is the full answer

$\lambda = 3$: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **A1**

$\lambda = -1$: $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **A1**

$\mathbf{x} = Ae^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + Be^{-t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **A2**

Question 9 — 3 marks

Answer: 0.3, 0.2; unstable node

$\lambda^2 - 0.5\lambda + 0.06 = 0$ **M1**

$\lambda = 0.3,\ 0.2$ **A1**

unstable node **A1**

Question 10 — 3 marks

Answer: $(0.99,\ -0.2)$

step 1: $(1,\ -0.1)$ **A1**

step 2: $(0.99,\ -0.2)$ **A2**

Every solution is also on ibmathrevision.com/hlai/phase-portraits-exam-practice
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