Markov Chains: exam practice

10 questions · 35 marks · worked solutions and markschemes

Name _____ Date _____ Score / 35

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlai/markov-chains-exam-practice

1. **3 marks · Paper 1 style · calculator allowed**

A Markov chain has the transition matrix $T = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$. The initial state vector is $S_0 = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$.

Calculate the state vector after one transition, S_1.

2. **2 marks · Paper 1 style · calculator allowed**

In a Markov chain the probability of moving from state A to state B is 0.35.

Write down the probability of staying in A and explain why each column of a transition matrix sums to 1. **[2]**

3. **3 marks · Paper 2 style · calculator allowed**

The weather is sunny (S) or rainy (R). After a sunny day the next is sunny with probability 0.75; after a rainy day the next is rainy with probability 0.6. Today is sunny.

Find the probability that it is sunny in 3 days' time. **[3]**

4. **3 marks · Paper 2 style · calculator allowed**

With the same weather model $\mathbf{T} = \begin{pmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{pmatrix}$:

Find the long-run proportion of sunny days. **[3]**

5. **3 marks · Paper 2 style · calculator allowed**

A car hire firm has offices in towns A and B. Each week 80% of cars at A stay at A, and 70% of cars at B stay at B. There are 60 cars at A and 40 at B.

Find the number of cars at each office after 2 weeks. **[3]**

6. **3 marks · Paper 1 style · calculator allowed**

A transition matrix is $\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$.

Find the steady-state vector. **[3]**

7. **4 marks · Paper 2 style · calculator allowed**

A Markov chain has two states, A and B. The transition matrix is $T = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$. The initial state matrix is $S_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$.

Find the state matrix after two transitions, S_2.

8. **3 marks · Paper 2 style · calculator allowed**

A mouse moves between rooms 1, 2 and 3 with transition matrix (columns are the current room)

$\begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.5 \end{pmatrix}$.

Find the long-run proportion of time spent in each room. **[3]**

9. **5 marks · Paper 2 style · calculator allowed**

Find the steady state probability vector for the transition matrix $T = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$.

10. **6 marks · Paper 2 style · calculator allowed**

A Markov chain models the choice of gym made each month by members of two gyms, A and B. The transition matrix is $T = \begin{pmatrix} 0.9 & a \\ 0.1 & b \end{pmatrix}$, where columns represent the current state and rows represent the next state.

1. Find the values of a and b, given that 15% of members who use gym B this month will use gym A next month.
2. Calculate the steady state probability vector for this Markov chain.

Answers and worked solutions

Question 1

3 marks

Answer: $(0.4, 0.6)$

$S_1 = TS_0 = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$. **M1**

Row 1:

$0.7(0.4) + 0.2(0.6) = 0.28 + 0.12 = 0.40$. **A1**

Row 2:

$0.3(0.4) + 0.8(0.6) = 0.12 + 0.48 = 0.60$.

$S_1 = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$. **A1**

Question 2

2 marks

Answer: 0.65

0.65 **A1**

from any state the system must move to some state **R1**

Question 3

3 marks

Answer: 0.632

$\mathbf{T} = \begin{pmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{pmatrix}$ **A1**

$\mathbf{T}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **M1**

0.632 **A1**

Question 4

3 marks

Answer: $\frac{8}{13}$

$0.25s = 0.4(1 - s)$ $M1$**A1**

$\frac{8}{13}$ **A1**

Question 5

3 marks

Answer: 60 at A, 40 at B

$\mathbf{T} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$ **A1**

$\mathbf{T}^2 \begin{pmatrix} 60 \\ 40 \end{pmatrix}$ **M1**

A: 60, B: 40 **A1**

Question 6

3 marks

Answer: $\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$

$0.1a = 0.3b$, $a + b = 1$ M**1A1**

$\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$ **A1**

Question 7

4 marks

Answer: $(0.565, 0.435)$

$S_2 = T^2 S_0$ **M1**

$T^2 = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix} \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}$ **A1**

$S_2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ **M1**

$= \begin{pmatrix} 0.565 \\ 0.435 \end{pmatrix}$ **A1**

Question 8

3 marks

Answer: 0.327, 0.388, 0.286

solve $\mathbf{Ts} = \mathbf{s}$, entries sum to 1 **M1**

room 1: 0.327, room 2: 0.388 **A1**

room 3: 0.286 **A1**

Question 9

5 marks

Answer: $(4/7, 3/7)$

Let the steady state vector be $\begin{pmatrix} x \\ y \end{pmatrix}$.

$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $x + y = 1$ **M1**

$0.7x + 0.4y = x \implies 0.4y = 0.3x \implies y = \frac{3}{4}x$ **A1**

Substitute into sum:

$x + 0.75x = 1 \implies 1.75x = 1$ **M1**

$x = \frac{1}{1.75} = \frac{4}{7}$ **A1**

$y = 1 - \frac{4}{7} = \frac{3}{7}$. Steady state vector is $\begin{pmatrix} 4/7 \\ 3/7 \end{pmatrix}$ **A1**

Question 10

6 marks

Answer: (a) 0.15; (b) 0.85; (Steady state vector) $(0.6, 0.4)$

1. From the prompt, $a = 0.15$. **A1**
   Columns must sum to 1, so $b = 1 - 0.15 = 0.85$. **A1**
2. Let steady state be $\begin{pmatrix} x \\ y \end{pmatrix}$.
   $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $x + y = 1$. **M1**
   $0.9x + 0.15y = x \implies 0.15y = 0.1x \implies x = 1.5y$. **M1**
   Substitute into sum:
   $1.5y + y = 1 \implies 2.5y = 1 \implies y = 0.4$. **A1**
   $x = 0.6$. Steady state vector is $\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$. **A1**

Every solution is also on ibmathrevision.com/hlai/markov-chains-exam-practice
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