IB MATHS AI HL · UNIT 4

Markov Chains and Transition Matrices: exam practice

10 questions · 37 marks · worked solutions and markschemes

Name _____ Date _____ Score / 37

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlai/markov-chains-and-transition-matrices-exam-practice

1. **4 marks · Paper 1 style · calculator allowed**

A system is modelled by the transition matrix $T = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$.

The initial state of the system is given by the column vector $s_0 = \begin{pmatrix} 100 \\ 50 \end{pmatrix}$.

Calculate the exact state vector after one transition, s_1.

2. **2 marks · Paper 1 style · calculator allowed**

$\mathbf{T} = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}$ and $\mathbf{s_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Find $\mathbf{s_1}$ and $\mathbf{s_2}$. **[2]**

3. 2 marks · Paper 1 style · calculator allowed

The steady state of $\begin{pmatrix} 1-a & b \\ a & 1-b \end{pmatrix}$ satisfies $as_1 = bs_2$.

Find the steady state when $a = 0.2$ and $b = 0.3$. **[2]**

4. 2 marks · Paper 2 style · calculator allowed

A gym member attends (A) or skips (K) each week. $\mathbf{T} = \begin{pmatrix} 0.8 & 0.35 \\ 0.2 & 0.65 \end{pmatrix}$, columns for this week.

A member attends this week. Find the probability they attend in 2 weeks' time. **[2]**

5. 3 marks · Paper 2 style · calculator allowed

A student is either on time (O) or late (L). If on time, the probability of being on time the next day is 0.9; if late, it is 0.5.

Write the transition matrix and find the long-run proportion of late days. **[3]**

6. 3 marks · Paper 2 style · calculator allowed

A transition matrix has eigenvalues 1 and 0.4, with eigenvector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ for $\lambda = 1$.

Write down the steady-state probabilities and explain why the chain approaches them. **[3]**

7. **4 marks · Paper 2 style · calculator allowed**

Customers buy brand X or brand Y. Of X buyers, 70% buy X again next month; of Y buyers, 40% switch to X. Initially 50% buy each.

Find the proportion buying X after 2 months and in the long run. **[4]**

8. **5 marks · Paper 1 style · calculator allowed**

A car rental company operates in two cities, City X and City Y. - A car rented in City X has a 70% chance of being returned to City X, and a 30% chance of being returned to City Y. - A car rented in City Y has a 90% chance of being returned to City Y, and a 10% chance of being returned to City X.
Currently, there are 400 cars in City X and 200 cars in City Y.
Using your GDC, find the number of cars expected to be in City X after 5 rental cycles (transitions). Round your answer to the nearest whole car.

9. **4 marks · Paper 2 style · calculator allowed**

A town's population moves between the centre (C) and the suburbs (S). Each year 15% of C moves to S and 5% of S moves to C. Now $40\,000$ live in C and $60\,000$ in S.

Find the population of C after 5 years and in the long run. **[4]**

10. **8 marks · Paper 1 style · calculator allowed**

A transition matrix has an unknown probability k:

$$T = \begin{pmatrix} 0.4 & k \\ 0.6 & 1-k \end{pmatrix}$$

The long-term steady state probability of being in state 1 is exactly 0.25.

Find the exact value of k.

Answers and worked solutions

Question 1 — 4 marks

Answer: $(80, 70)$

The state after one transition is $\boldsymbol{s}_1 = T\boldsymbol{s}_0$. **[M1]**

$\boldsymbol{s}_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 100 \\ 50 \end{pmatrix}$. **[M1]**

Row 1:

$(0.7 \times 100) + (0.2 \times 50) = 70 + 10 = 80$. **[A1]**

Row 2:

$(0.3 \times 100) + (0.8 \times 50) = 30 + 40 = 70$. **[A1]**

$\boldsymbol{s}_1 = \begin{pmatrix} 80 \\ 70 \end{pmatrix}$.

Question 2 — 2 marks

Answer: $\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$, $\begin{pmatrix} 0.44 \\ 0.56 \end{pmatrix}$

$\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$ **A1**

$\begin{pmatrix} 0.44 \\ 0.56 \end{pmatrix}$ **A1**

Question 3 — 2 marks

Answer: $\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$

$0.2s_1 = 0.3(1 - s_1)$ **M1**

$\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$ **A1**

Question 4 — 2 marks

Answer: $\frac{71}{100}$

$0.8(0.8) + 0.2(0.35)$ **M1**

0.71 **A1**

Question 5 — 3 marks

Answer: $\frac{1}{6}$

$\begin{pmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{pmatrix}$ **A1**

$0.1o = 0.5(1 - o)$ **M1**

late $\frac{1}{6}$ **A1**

Question 6 — 3 marks

Answer: $\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$

$\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$ **A2**

$0.4^n \to 0$ **R1**

Question 7 — 4 marks

Answer: 0.565; $\frac{4}{7}$

$\mathbf{T} = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$ **A1**

$\mathbf{T}^2 \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$: 0.565 **A1**

$0.3x = 0.4(1 - x)$ **M1**

$\frac{4}{7}$ **A1**

Question 8 — 5 marks

Answer: 169

Transition matrix (columns = from X, Y): $T = \begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}$. **[A1]**

Initial state $\boldsymbol{s}_0 = \begin{pmatrix} 400 \\ 200 \end{pmatrix}$. **[A1]**

We need $\boldsymbol{s}_5 = T^5\boldsymbol{s}_0$. **[M1]**

Using GDC:

$\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}^5 \begin{pmatrix} 400 \\ 200 \end{pmatrix} = \begin{pmatrix} 169.44 \\ 430.56 \end{pmatrix}$. **[M1]**

Number of cars in City $X \approx 169$ cars. **[A1]**

Question 9 — 4 marks

Answer: $29\,900$; $25\,000$

$\mathbf{T} = \begin{pmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{pmatrix}$ **A1**

$\mathbf{T}^5\mathbf{s}_0$ **M1**

$29\,900$ **A1**

long run $25\,000$ **A1**

Question 10 — 8 marks

Answer: 0.2

Let steady state be $\boldsymbol{s} = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$ (since components must sum to 1). **[M1][A1]**

$T\boldsymbol{s} = \boldsymbol{s} \implies \begin{pmatrix} 0.4 & k \\ 0.6 & 1-k \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$. **[M1]**

Read the first row equation:

$(0.4 \times 0.25) + (k \times 0.75) = 0.25$. **[M1][A1]**

$0.10 + 0.75k = 0.25$. **[A1]**

$0.75k = 0.15$. **[M1]**

$k = 0.15/0.75 = 1/5 = 0.2$. **[A1]**

Every solution is also on ibmathrevision.com/hlai/markov-chains-and-transition-matrices-exam-practice
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