Linear Combinations of Random Variables & The Central Limit Theorem: exam practice

10 questions · 38 marks · worked solutions and markschemes

Name _____ Date _____ Score / 38

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlai/linear-combinations-of-random-variables-and-the-central-limit-theorem-exam-practice

1. **3 marks · Paper 1 style · calculator allowed**

The weights of regular apples A are normally distributed with variance 15. The weights of large apples B are normally distributed with variance 25. An order contains 4 independently chosen regular apples and 2 independently chosen large apples. Find the exact variance of the total combined weight of the order.

2. **3 marks · Paper 1 style · calculator allowed**

The volumes of small cartons of juice X are normally distributed with variance 8. The volumes of large cartons Y are normally distributed with variance 12. Find the exact variance of $D = X_1 + X_2 + X_3 - Y$, where X_1, X_2, X_3 are three independently chosen small cartons and Y is an independently chosen large carton.

3. **3 marks · Paper 2 style · calculator allowed**

$X \sim N(20,\ 3^2)$ and $Y \sim N(15,\ 4^2)$ are independent.

Find the distribution of $X + Y$ and $P(X + Y > 40)$. **[3]**

4. **3 marks · Paper 2 style · calculator allowed**

The time to serve a customer has mean 3.2 min and standard deviation 1.5 min. A sample of 50 customers is taken.

Use the central limit theorem to find the probability that the mean time exceeds 3.5 min. **[3]**

5. **3 marks · Paper 1 style · calculator allowed**

$E(X) = 5$, $\mathrm{Var}(X) = 2$, $E(Y) = 3$, $\mathrm{Var}(Y) = 4$, with X and Y independent.

Find $E(3X - 2Y)$ and $\mathrm{Var}(3X - 2Y)$. **[3]**

6. **4 marks · Paper 2 style · calculator allowed**

The mass of an apple is $N(150,\ 12^2)$ g. A bag holds 6 apples, and the empty bag has mass 20 g.

Find the mean and standard deviation of the total mass, and the probability it exceeds 950 g. **[4]**

7. **4 marks · Paper 1 style · calculator allowed**

A random variable X follows a normal distribution $X \sim N(50, 4^2)$. A new variable Y is defined by the linear transformation $Y = 3X - 10$. Find the exact expected value and variance of Y.

8. **4 marks · Paper 1 style · calculator allowed**

The time T, in minutes, that a plumber spends on a repair is a random variable with $E(T) = 25$ and $\text{Var}(T) = 16$. The plumber charges €0.5 per minute plus a fixed call-out fee of €4, so the charge in euros is $C = 0.5T + 4$. Find the exact expected value and variance of C.

9. **4 marks · Paper 1 style · calculator allowed**

Two independent random variables are distributed as $X \sim N(12, 3^2)$ and $Y \sim N(8, 2^2)$. Let $W = 2X - 3Y + 5$. Find the exact expected value and variance of W.

10. **7 marks · Paper 2 style · calculator allowed**

The mass of a chocolate bar, X g, is distributed as $X \sim N(45, 2^2)$. A gift box contains 6 randomly chosen bars, and the empty packaging has mass P g, where $P \sim N(20, 1.5^2)$. All masses are independent.

1. Find the probability that the total mass of a full gift box is between 285 g and 295 g.
2. Two full gift boxes are chosen at random. Find the probability that the first box is more than 5 g heavier than the second box.

Answers and worked solutions

Question 1 — 3 marks

Answer: 110

Write the total as a sum of independent items: $W = A_1 + A_2 + A_3 + A_4 + B_1 + B_2$. Since the apples are independent, their variances add (not $4^2\text{Var}(A)$). **R1**

Add the variances:

$\text{Var}(W) = 4\text{Var}(A) + 2\text{Var}(B) = 4(15) + 2(25)$ **M1**

$\text{Var}(W) = 110$ **A1**

Question 2 — 3 marks

Answer: 36

Since the cartons are independent, variances add, even for the subtracted term: $\text{Var}(-Y) = (-1)^2\text{Var}(Y)$ (and $\text{Var}(X_1 + X_2 + X_3) = 3\text{Var}(X)$, not $3^2\text{Var}(X)$). **R1**

Add the variances:

$\text{Var}(D) = 3\text{Var}(X) + \text{Var}(Y) = 3(8) + 12$ **M1**

$\text{Var}(D) = 36$ **A1**

Question 3 — 3 marks

Answer: $N(35,\ 5^2)$; 0.159

$N(35,\ 25)$ **A1A1**

0.159 **A1**

Question 4 — 3 marks

Answer: 0.0786

$\bar{T} \approx N\left(3.2,\ \dfrac{1.5^2}{50}\right)$ **A1**

$P(\bar{T} > 3.5)$ **M1**

0.0786 **A1**

Question 5 — 3 marks

Answer: 9; 34

$15 - 6 = 9$ **A1**

$9(2) + 4(4)$ **M1**

34 **A1**

Question 6 — 4 marks

Answer: $920\ g$, $29.4\ g$; 0.154

mean $920\ g$ **A1**

$\sigma = 12\sqrt{6} = 29.4\ g$ **A1**

$P(T > 950)$ **M1**

0.154 **A1**

Question 7 — 4 marks

Answer: ($E(Y)$ 140; ($\text{Var}(Y)$ 144

Apply the expectation rule:

$E(Y) = 3E(X) - 10 = 3(50) - 10$ **M1**

$E(Y) = 140$ **A1**

Apply the variance rule (the constant does not affect variance):

$\text{Var}(Y) = 3^2\text{Var}(X) = 9 \times 4^2$ **M1**

$\text{Var}(Y) = 144$ **A1**

Question 8 — 4 marks

Answer: ($E(C)$ (euros) 16.5; ($\text{Var}(C)$ 4

Apply the expectation rule:

$E(C) = 0.5E(T) + 4 = 0.5(25) + 4$ **M1**

$E(C) = 16.5$ (euros) **A1**

Apply the variance rule (the constant does not affect variance):

$\text{Var}(C) = 0.5^2\text{Var}(T) = 0.25 \times 16$ **M1**

$\text{Var}(C) = 4$ **A1**

Question 9 — 4 marks

Answer: ($E(W)$ 5; ($\text{Var}(W)$ 72

Apply the expectation rule:

$E(W) = 2E(X) - 3E(Y) + 5 = 2(12) - 3(8) + 5$ **M1**

$E(W) = 5$ **A1**

Apply the variance rule (variances add even when the variables are subtracted, and the constant has no effect):

$\text{Var}(W) = 2^2(3^2) + (-3)^2(2^2) = 36 + 36$ **M1**

$\text{Var}(W) = 72$ **A1**

Question 10 — 7 marks

Answer: (a) 0.671; (b) 0.245

1. Let $T = X_1 + \cdots + X_6 + P$. $E(T) = 6(45) + 20 = 290$. **[A1]**
   $Var(T) = 6(2^2) + 1.5^2 = 26.25$ **[M1][A1]**
   $P(285 < T < 295) \approx 0.671$ (3 s.f.) (using $SD = \sqrt{26.25} \approx 5.12$). **[A1]**
2. Let $D = T_1 - T_2$.
   $E(D) = 0$,
   $Var(D) = 26.25 + 26.25 = 52.5$. **[M1][A1]**
   $P(D > 5) \approx 0.245$ (3 s.f.) (using $SD = \sqrt{52.5} \approx 7.25$). **[A1]**

Every solution is also on ibmathrevision.com/hlai/linear-combinations-of-random-variables-and-the-central-limit-theorem-exam-practice
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