# Graph Theory: exam practice

10 questions · 31 marks · worked solutions and markschemes

Name _____ Date _____ Score / 31

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlai/graph-theory-exam-practice

**1.** **2 marks · Paper 1 style · calculator allowed**

A complete graph $K_n$ has $n$ vertices where every vertex is connected to every other vertex by exactly one edge. Calculate the exact number of edges in a complete graph $K_6$.

**2.** **3 marks · Paper 1 style · calculator allowed**

The adjacency matrix $A$ for a graph with 3 vertices is $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Calculate the exact number of walks of length 3 that start and end at vertex 1.

**3.** **2 marks · Paper 1 style · calculator allowed**

A graph has vertices $A$, $B$, $C$, $D$ with weighted adjacency: $AB = 3$, $AC = 7$, $BC = 2$, $BD = 6$, $CD = 1$.

Find the shortest distance from $A$ to $D$. **[2]**

**4.** **2 marks · Paper 1 style · calculator allowed**

A connected graph has vertex degrees $2, 3, 3, 4, 2$.

State, with a reason, whether the graph has an Eulerian circuit, an Eulerian trail, or neither. **[2]**

**5.** **3 marks · Paper 2 style · calculator allowed**

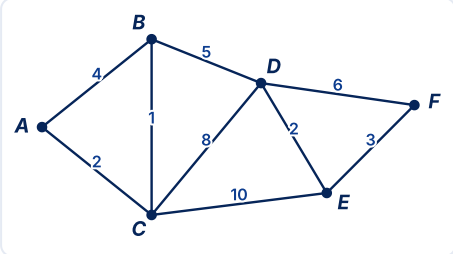
Use Kruskal's algorithm to find the minimum spanning tree of the graph with edges $PQ = 7$, $PR = 5$, $QR = 4$, $QS = 6$, $RS = 9$, $RT = 8$, $ST = 3$.

Find the total weight of the minimum spanning tree. **[3]**

**6.** **4 marks · Paper 2 style · calculator allowed**

A weighted graph has edges $AB = 4$, $AC = 2$, $BC = 1$, $BD = 5$, $CD = 8$, $CE = 10$, $DE = 2$, $DF = 6$, $EF = 3$.

Use Dijkstra's algorithm to find the shortest route from $A$ to $F$ and its length. **[4]**

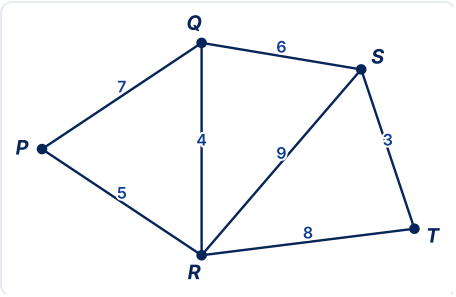



Find the shortest route from A to F

**7.** **4 marks · Paper 2 style · calculator allowed**

A road network has edges (lengths in km) $PQ = 7$, $PR = 5$, $QR = 4$, $QS = 6$, $RS = 9$, $RT = 8$, $ST = 3$. A gritter must drive along every road and return to the start.

Find the length of the shortest route. **[4]**




Road lengths in km

**8.** **4 marks · Paper 2 style · calculator allowed**

A complete graph on $A$, $B$, $C$, $D$ has weights $AB = 5$, $AC = 8$, $AD = 6$, $BC = 4$, $BD = 7$, $CD = 3$.

Use the nearest-neighbour algorithm from $A$ to find an upper bound for the travelling salesman problem, and find the optimal tour length. **[4]**

**9.** **5 marks · Paper 2 style · calculator allowed**

A courier must visit 5 locations: A, B, C, D, and E. The graph connecting these locations is fully connected. The lengths of the edges connected to vertex A are: AB=4, AC=5, AD=8, and AE=6. The minimum spanning tree for the subgraph containing only vertices B, C, D, and E has a total weight of 14.

Use the deleted vertex algorithm (deleting vertex A) to find a lower bound for the Travelling Salesman Problem for this network.

**10.** **2 marks · Paper 1 style · calculator allowed**

A water company's network has $11$ junctions joined by $17$ pipes, and every junction can be reached from every other. To save money, the company keeps open only the pipes of a minimum spanning tree.

a. Write down the number of pipes that stay open. **[1]**

b. Write down the number of pipes that are closed. **[1]**

## Answers and worked solutions

### Question 1 — 2 marks

**Answer:** $15$

Each of the $n$ vertices joins the other $n-1$ vertices and each edge is counted twice, so the number of edges is $\frac{n(n-1)}{2}$. **M1**

For $K_6$: $\frac{6\times5}{2}=15$ edges. **A1**

(Alternative: $5+4+3+2+1=15$.)

### Question 2 — 3 marks

**Answer:** $2$

Walks of length 3 from vertex 1 to vertex 1 are given by the (1,1) entry of $A^3$ **M1**

$A^2=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$; (1,1) entry of $A^3=A^2A$:

$(2)(0)+(1)(1)+(1)(1)$ **(M1)**

Number of walks $=2$ **A1**

### Question 3 — 2 marks

**Answer:** $6$

$A\to B\to C\to D$ **M1**

$6$ **A1**

### Question 4 — 2 marks

**Answer:** an Eulerian trail

exactly two odd vertices **R1**

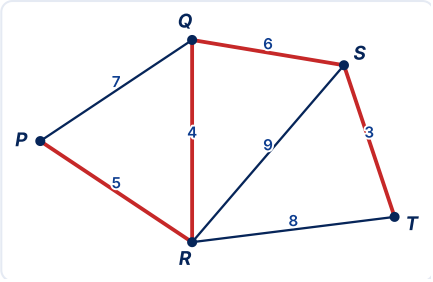
Eulerian trail (not a circuit) **A1**

### Question 5 — 3 marks

**Answer:** $18$

choose $ST=3$, $QR=4$, $PR=5$, $QS=6$ $M1$**A1**

$18$ **A1**




Kruskal: ST, QR, PR, QS; total 18

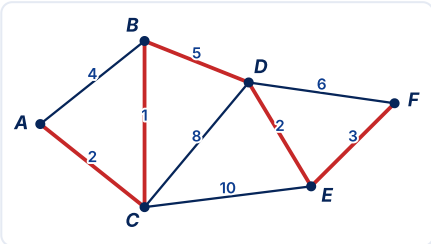
### Question 6 — 4 marks

**Answer:** $13$, via $ACBDEF$

labels $C=2$, $B=3$, $D=8$, $E=10$ $M1$**A1**

$F=13$ **A1**

$A\to C\to B\to D\to E\to F$ **A1**




Shortest route ACBDEF, length 13

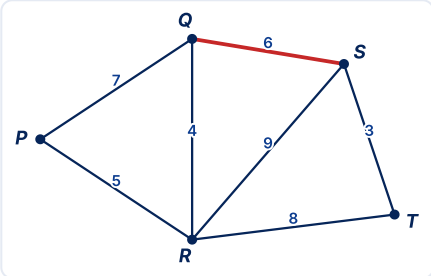
### Question 7 — 4 marks

**Answer:** $48$ km

odd vertices $Q$ and $S$ **A1**

shortest $Q$ to $S$: $QS=6$ $M1$**A1**

$42+6=48$ km **A1**




Repeat QS: $42+6=48$ km

### Question 8 — 4 marks

**Answer:** upper bound $18$; optimal $18$

$A\to B\to C\to D\to A$ **M1**

$5+4+3+6=18$ **A1**

three distinct tours: $18$, $22$, $22$ **M1**

optimal $18$ **A1**

### Question 9 — 5 marks

**Answer:** $23$

Delete vertex $A$ and all edges connected to it. **M1**

The weight of the Minimum Spanning Tree of the remaining vertices $(B,C,D,E)$ is given as 14. **A1**

Identify the two edges with the lowest weight connecting vertex $A$ to the remaining network. **M1**

These edges are AB (4) and AC (5). **A1**

Lower bound = Weight of MST + sum of two lowest edges = $14+4+5=23$. **A1**

### Question 10 — 2 marks

**Answer:** (a) $10$; (b) $7$

a. a spanning tree on $11$ vertices has $11-1=10$ edges **A1**

b. $17-10=7$ **A1**

Every solution is also on ibmathrevision.com/hlai/graph-theory-exam-practice

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