Eigenvalues, Eigenvectors and Markov Chains: exam practice

10 questions · 41 marks · worked solutions and markschemes

Name _____ Date _____ Score / 41

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlai/eigenvalues-eigenvectors-and-markov-chains-exam-practice

1. **3 marks · Paper 2 style · calculator allowed**

Customers choose between three brands X, Y and Z each month with transition matrix (columns are this month) $\mathbf{T} = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$.

Find the long-term market share of each brand. **[3]**

2. **3 marks · Paper 1 style · calculator allowed**

Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ with eigenvalues 2 and 3.

Find an eigenvector for $\lambda = 3$ and one for $\lambda = 2$. **[3]**

3. **4 marks · Paper 1 style · calculator allowed**

Find the exact eigenvalues of the matrix $M = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$.

4. **6 marks · Paper 1 style · calculator allowed**

A transition matrix is given by $T = \begin{pmatrix} 0.6 & k \\ 0.4 & 1-k \end{pmatrix}$, where k is a constant and $0 < k < 1$.

The long-term steady-state probability for the first state is exactly $\frac{2}{3}$.

Calculate the exact value of k.

5. **3 marks · Paper 1 style · calculator allowed**

A Markov chain has a transition matrix $T = \begin{pmatrix} 0.5 & a \\ 0.5 & 0.2 \end{pmatrix}$. State the value of a, and write down the exact dominant eigenvalue for this matrix.

6. **3 marks · Paper 2 style · calculator allowed**

A website has two pages. A visitor on page 1 moves to page 2 with probability 0.3; a visitor on page 2 moves to page 1 with probability 0.5. Otherwise they stay.

(a) Write down the transition matrix. **[1]**

(b) A visitor starts on page 1. Find the probability that after 3 moves they are on page 2. **[2]**

7. **5 marks · Paper 1 style · calculator allowed**

Let $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

(a) Find the eigenvalues of $\mathbf{A}$. **[3]**

(b) Find an eigenvector for each eigenvalue. **[2]**

8. **5 marks · Paper 2 style · calculator allowed**

Each day a commuter travels by bus (B) or train (T). If she takes the bus one day, the probability she takes the bus the next day is 0.6. If she takes the train, the probability she takes the train the next day is 0.8.

(a) Write down the transition matrix $\mathbf{T}$, with columns for today's choice. **[2]**

(b) Find the long-term proportion of days she takes the bus. **[3]**

9. **5 marks · Paper 2 style · calculator allowed**

A transition matrix is $\mathbf{T} = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$ and the initial state is $\mathbf{s}_0 = \begin{pmatrix} 100 \\ 200 \end{pmatrix}$.

(a) Find $\mathbf{T}^3\mathbf{s}_0$. **[2]**

(b) Find the steady-state vector for this population of 300. **[3]**

10. **4 marks · Paper 1 style · calculator allowed**

Let $\mathbf{M} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$.

(a) Show that the eigenvalues of $\mathbf{M}$ are 1 and 0.7. **[2]**

(b) Explain what happens to $\mathbf{M}^n$ as $n \to \infty$, in terms of the eigenvalues. **[2]**

Answers and worked solutions

Question 1

3 marks

Answer: each $\frac{1}{3}$

solve $\mathbf{Ts} = \mathbf{s}$ with entries summing to 1 M1

X: $\frac{1}{3}$ A1

Y: $\frac{1}{3}$, Z: $\frac{1}{3}$ A1

Question 2

3 marks

Answer: a proof: the worked solution above is the full answer

$\lambda = 3$: $-x = 0$, so $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $M1$A1

$\lambda = 2$: $x + y = 0$, so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ A1

Question 3

4 marks

Answer: $2, 5$

Set up the characteristic equation $\det(M - \lambda I) = 0$:

$\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$ M1

Expand the determinant:

$(4-\lambda)(3-\lambda) - (2)(1) = 0$ (M1)

$\lambda^2 - 7\lambda + 10 = 0$ A1

Factorise:

$(\lambda - 5)(\lambda - 2) = 0 \implies \lambda = 2, 5$ A1

Question 4

6 marks

Answer: 0.8

Set up the steady-state equation $T\mathbf{x} = \mathbf{x}$ with $\mathbf{x} = \begin{pmatrix} p \\ q \end{pmatrix}$:

$\begin{pmatrix} 0.6 & k \\ 0.4 & 1-k \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$ M1

Use $p = \frac{2}{3}$ and $p + q = 1$: $q = \frac{1}{3}$ A1

Substitute into the top row:

$0.6\left(\frac{2}{3}\right) + k\left(\frac{1}{3}\right) = \frac{2}{3}$ M1

Multiply by 3: $0.6(2) + k = 2$ M1

$1.2 + k = 2$ A1

$k = 0.8$ $\left(= \frac{4}{5}\right)$, which satisfies $0 < k < 1$ A1

Question 5

3 marks

Answer: (a) 0.8; (dominant eigenvalue) 1

Each column of a transition matrix sums to 1: $a + 0.2 = 1$ (M1)

$a = 0.8$ A1

A transition matrix always has eigenvalue 1, and this is the dominant eigenvalue: $\lambda = 1$ A1

Question 6

3 marks

Answer: (b) 0.372

(a)

$\begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix}$ A1

(b)

$\mathbf{T}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ M1

0.372 A1

Question 7

5 marks

Answer: (a) $\lambda = 2$, $\lambda = 5$

(a)

$\det(\mathbf{A} - \lambda\mathbf{I}) = (4-\lambda)(3-\lambda) - 2$ M1

$\lambda^2 - 7\lambda + 10 = 0$ A1

$\lambda = 2,\ 5$ A1

(b)

$\lambda = 5$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ A1

$\lambda = 2$: $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ A1

Question 8

5 marks

Answer: (b) $\frac{1}{3}$

(a)

$\begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}$ A2

(b)

steady state $\mathbf{Ts} = \mathbf{s}$ M1

$0.4b = 0.2(1 - b)$ A1

$b = \frac{1}{3}$ A1

Question 9

5 marks

Answer: (a) $\begin{pmatrix} 169.5 \\ 130.5 \end{pmatrix}$; (b) $\begin{pmatrix} 171.4 \\ 128.6 \end{pmatrix}$

(a)

GDC (M1)

$\begin{pmatrix} 169.5 \\ 130.5 \end{pmatrix}$ A1

(b)

$0.3x = 0.4y$, $x + y = 300$ $M1$A1

$\begin{pmatrix} 171.4 \\ 128.6 \end{pmatrix}$ A1

Question 10

4 marks

Answer: a proof: the worked solution above is the full answer

(a)

$\lambda^2 - 1.7\lambda + 0.7 = 0$ M1

$(\lambda - 1)(\lambda - 0.7) = 0$ A1 AG

(b)

$0.7^n \to 0$ R1

$\mathbf{M}^n$ tends to a matrix whose columns are both the steady state $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ A1

Every solution is also on ibmathrevision.com/hlai/eigenvalues-eigenvectors-and-markov-chains-exam-practice
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