Differential Equations: exam practice

10 questions · 35 marks · worked solutions and markschemes

Name _____ Date _____ Score / 35

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlai/differential-equations-exam-practice

1. **2 marks · Paper 1 style · calculator allowed**

The slope field of $\frac{dy}{dx} = y(2 - y)$ is drawn. A solution curve passes through $(0,\ 0.5)$.

Describe the long-term behaviour of this solution. **[2]**

2. **2 marks · Paper 1 style · calculator allowed**

A slope field for $\frac{dy}{dx} = x^2 - y$ is drawn.

Find the gradient at $(2, 1)$ and the equation of the curve on which the slopes are zero. **[2]**

3. **2 marks · Paper 1 style · calculator allowed**

The slope field of $\frac{dy}{dx} = -\frac{x}{y}$ is drawn.

Describe the shape of the solution curves and give the curve through $(3, 4)$. **[2]**

4. **3 marks · Paper 1 style · calculator allowed**

Solve $\frac{dy}{dx} = \frac{x}{y}$, $y(0) = 3$, $y > 0$.

Find y. **[3]**

5. **3 marks · Paper 2 style · calculator allowed**

A system is $\dot{x} = -2x + y$, $\dot{y} = x - 2y$.

Find the eigenvalues and classify the equilibrium point. **[3]**

6. **4 marks · Paper 1 style · calculator allowed**

A slope field is drawn for $\frac{dy}{dx} = x + y$.

Find the gradient of the slope field at $(1, 2)$, $(-1, 1)$ and $(0, -3)$, and describe where the slopes are zero. **[4]**

7. **5 marks · Paper 2 style · calculator allowed**

Consider the differential equation $\frac{dy}{dx} = x + y^2$ with the initial condition $y(0) = 1$.

Use Euler's method with a step size of $h = 0.1$ to find an approximate value for $y(0.2)$.

8. **6 marks · Paper 2 style · calculator allowed**

A predator-prey system is modelled by the coupled differential equations:

$\frac{dx}{dt} = 0.2x - 0.01xy$

$\frac{dy}{dt} = -0.1y + 0.002xy$

where x is the number of prey and y is the number of predators.

Find the non-zero equilibrium point of this system.

9. **4 marks · Paper 1 style · calculator allowed**

The rate of change of a population of bacteria, y, is directly proportional to its size, modelled by $\frac{dy}{dt} = ky$.

1. Given that $y(0) = 50$, solve the differential equation to find y in terms of k and t.
2. If $y(2) = 100$, find the exact value of k.

10. **4 marks · Paper 1 style · calculator allowed**

The growth of a population of insects, $P(t)$, is modelled by the logistic differential equation $\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{1000}\right)$.

1. State the carrying capacity of this population.
2. Find the exact population size at which the population is growing at its fastest rate.

Answers and worked solutions

Question 1 — 2 marks

Answer: $y \to 2$

$\frac{dy}{dx} > 0$ for $0 < y < 2$ **R1**

$y \to 2$ **A1**

Question 2 — 2 marks

Answer: 3; $y = x^2$

3 **A1**

$y = x^2$ **A1**

Question 3 — 2 marks

Answer: circles; $x^2 + y^2 = 25$

circles centred at the origin **A1**

$x^2 + y^2 = 25$ **A1**

Question 4 — 3 marks

Answer: $y = \sqrt{x^2 + 9}$

$\int y\,dy = \int x\,dx$ **M1**

$y^2 = x^2 + 9$ **A1**

$y = \sqrt{x^2 + 9}$ **A1**

Question 5 — 3 marks

Answer: $\lambda = -1, -3$; stable node

$(\lambda + 2)^2 - 1 = 0$ **M1**

$\lambda = -1,\ -3$ **A1**

stable node **A1**

Question 6 — 4 marks

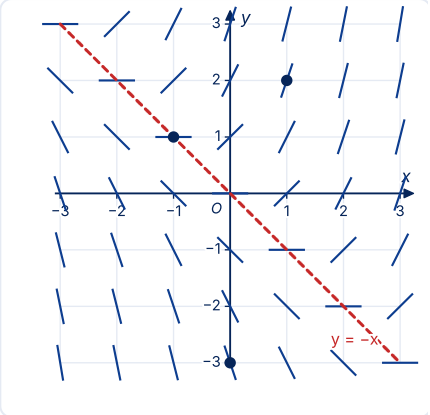
Answer: 3, 0, -3; on $y = -x$

3 **A1**

0 **A1**

-3 **A1**

along the line $y = -x$ **A1**



Slopes are zero on $y = -x$

Question 7 — 5 marks

Answer: 1.231

Start: $x_0 = 0, y_0 = 1$.

First step:

$y_1 = y_0 + h(x_0 + y_0^2) = 1 + 0.1(0 + 1^2)$ **M1**

$y_1 = 1.1$, at $x_1 = 0.1$ **A1**

Second step:

$y_2 = y_1 + h(x_1 + y_1^2) = 1.1 + 0.1(0.1 + 1.1^2)$ **M1**

$= 1.1 + 0.1(1.31)$ **A1**

$y(0.2) \approx 1.231$ **A1**

Question 8 — 6 marks

Answer: $(50, 20)$

At equilibrium, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. **[M1]**

$0.2x - 0.01xy = 0 \implies x(0.2 - 0.01y) = 0$ **[M1]**

$-0.1y + 0.002xy = 0 \implies y(-0.1 + 0.002x) = 0$ **[M1]**

For non-zero equilibrium ($x \neq 0, y \neq 0$):

$0.2 - 0.01y = 0 \implies 0.01y = 0.2 \implies y = 20$ **[A1]**

$-0.1 + 0.002x = 0 \implies 0.002x = 0.1 \implies x = 50$ **[A1]**

The equilibrium point is $(50, 20)$. **[A1]**

Question 9 — 4 marks

Answer: (a) $y)$ $50e^{(kt)}$; (b) $k)$ $\ln(2)/2$

1. Separating variables: $\int \frac{1}{y}\,dy = \int k\,dt$. **[M1]**
   $\ln|y| = kt + C \implies y = Ae^{kt}$.
   Since $y(0) = 50$, $A = 50$, so $y = 50e^{kt}$. **[A1]**
2. Substitute $t = 2, y = 100$: $100 = 50e^{2k}$. **[M1]**
   $e^{2k} = 2 \implies 2k = \ln(2)$.
   $k = \frac{1}{2}\ln(2)$ (or $\ln(\sqrt{2})$). **[A1]**

Question 10 — 4 marks

Answer: (Carrying capacity) 1000; (Population at the fastest growth) 500

1. Carrying capacity $K = 1000$ **A1**
2. Expand the growth rate: $\frac{dP}{dt} = 0.05P - 0.00005P^2$
   Differentiate with respect to P and set equal to zero: $0.05 - 0.0001P = 0$ **M1**
   A1
   $P = 500$ (half the carrying capacity) **A1**

Every solution is also on ibmathrevision.com/hlai/differential-equations-exam-practice
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