Differential Equations and Eulers Method: exam practice

10 questions · 45 marks · worked solutions and markschemes

Name _____ Date _____ Score / 45

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlai/differential-equations-and-eulers-method-exam-practice

1. **4 marks · Paper 2 style · calculator allowed**

Solve $\frac{dy}{dx} - \frac{y}{x} = x^2$, $x > 0$, $y(1) = 2$.

Find y. **[4]**

2. **4 marks · Paper 2 style · calculator allowed**

Use Euler's method with $h = 0.1$ for the coupled system $\dot{x} = 0.5x - 0.1xy$, $\dot{y} = -0.3y + 0.05xy$, with $x(0) = 4$, $y(0) = 2$, to estimate $x(0.2)$ and $y(0.2)$.

Find the estimates. **[4]**

3. **4 marks · Paper 1 style · calculator allowed**

Find the integrating factor for $\frac{dy}{dx} + \frac{3y}{x} = 1$, $x > 0$, and solve the equation given $y(1) = 0$.

Find y. **[4]**

4. **5 marks · Paper 1 style · calculator allowed**

The rate at which a hot liquid cools is proportional to its temperature y, modelled by $\frac{dy}{dt} = -ky$ for a positive constant k. Initially, $y(0) = 100$, and after 2 minutes, $y(2) = 50$. Find the exact temperature of the liquid after 6 minutes, $y(6)$.

5. **5 marks · Paper 1 style · calculator allowed**

A system of coupled differential equations is given by $\frac{dx}{dt} = x + y$ and $\frac{dy}{dt} = x - y$. At $t = 0$, the initial coordinates are $(x_0, y_0) = (1, 1)$. Use Euler's method with a step size of $h = 0.5$ to find the coordinates (x, y) at $t = 1$.

6. **5 marks · Paper 1 style · calculator allowed**

Consider the differential equation $\frac{dy}{dx} = x - y^2$, with initial condition $y(1) = 1$.
Use Euler's method with a step size of $h = 0.2$ to find the approximate value of $y(1.4)$. Show your intermediate steps in a table.

7. **6 marks · Paper 1 style · calculator allowed**

Find the exact particular solution to the differential equation:

$$\frac{dy}{dx} = y\cos x$$

given the boundary condition that $y(0) = e^2$.

8. **3 marks · Paper 2 style · calculator allowed**

The equation $\ddot{x} + 2\dot{x} + 5x = 0$ is written as $\dot{x} = y$, $\dot{y} = -5x - 2y$.

Find the eigenvalues and describe the motion. **[3]**

9. **4 marks · Paper 2 style · calculator allowed**

The equation $\ddot{x} + 3\dot{x} + 2x = 0$ models a damped spring.

Write it as a system and find the eigenvalues; describe the motion. **[4]**

10. **5 marks · Paper 2 style · calculator allowed**

Solve $\dfrac{dy}{dx} + 2y = e^{x}$, $y(0) = 1$, using an integrating factor.

Find y. **[5]**

Answers and worked solutions

Question 1

4 marks

Answer: $y = \dfrac{x^3 + 3x}{2}$

integrating factor $\dfrac{1}{x}$ A1

$\dfrac{d}{dx}\left(\dfrac{y}{x}\right) = x$ M1

$\dfrac{y}{x} = \dfrac{x^2}{2} + \dfrac{3}{2}$ A1

$y = \dfrac{x^3}{2} + \dfrac{3x}{2}$ A1

Question 2

4 marks

Answer: $x \approx 4.24$, $y \approx 1.96$

$x_1 = 4.12$, $y_1 = 1.98$ $M1$A1

$x_2 = 4.24$ A1

$y_2 = 1.96$ A1

Question 3

4 marks

Answer: $y = \dfrac{x}{4} - \dfrac{1}{4x^3}$

$e^{\int \frac{3}{x} dx} = x^3$ A1

$\dfrac{d}{dx}(x^3 y) = x^3$ M1

$x^3 y = \dfrac{x^4}{4} - \dfrac{1}{4}$ A1

$y = \dfrac{x}{4} - \dfrac{1}{4x^3}$ A1

Question 4

5 marks

Answer: 12.5

Use the exponential decay solution with $y(0) = 100$: $y(t) = 100e^{-kt}$ M1

Use $y(2) = 50$: $100e^{-2k} = 50$ M1

So the 2-minute decay factor is $e^{-2k} = 0.5$ A1

Write $y(6)$ in terms of this factor:

$y(6) = 100e^{-6k} = 100\left(e^{-2k}\right)^3$ M1

$y(6) = 100(0.5)^3 = 12.5$ A1

Question 5

5 marks

Answer: $(3.5, 1.5)$

Apply Euler's method to both equations with $h = 0.5$:

$x_1 = x_0 + h(x_0 + y_0) = 1 + 0.5(1 + 1)$,

$y_1 = y_0 + h(x_0 - y_0) = 1 + 0.5(1 - 1)$ M1

Coordinates at $t = 0.5$:

$(x_1, y_1) = (2, 1)$ A1

Second step using the new coordinates: $x_2 = 2 + 0.5(2 + 1)$,

$y_2 = 1 + 0.5(2 - 1)$ M1

$x_2 = 3.5$ A1

$y_2 = 1.5$, so at $t = 1$,

$(x, y) = (3.5, 1.5)$ A1

Question 6

5 marks

Answer: 1.04

Euler's method:

$y_{n+1} = y_n + h\, f(x_n, y_n)$ with $f(x, y) = x - y^2$,

$x_0 = 1, y_0 = 1, h = 0.2$ M1

Step 1: $f(x_0, y_0) = 1 - 1^2 = 0$ A1

$y_1 = 1 + 0.2(0) = 1$ at $x_1 = 1.2$ A1

Step 2:

$f(x_1, y_1) = 1.2 - 1^2 = 0.2$ A1

$y_2 = 1 + 0.2(0.2) = 1.04$ at $x_2 = 1.4$ A1

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	0	1
1	1.2	1	0.2	1.04
2	1.4	1.04		

Approximate value: $y(1.4) \approx 1.04$

Question 7

6 marks

Answer: $e^{(sinx + 2)}$

Separate variables:

$\frac{1}{y}\, dy = \cos x\, dx$ M1

Integrate both sides:

$\int \frac{1}{y}\, dy = \int \cos x\, dx$

$\ln|y| = \sin x + C$ A1 A1

Substitute $x = 0, y = e^2$:

$\ln(e^2) = \sin 0 + C$ M1

$2 = 0 + C \implies C = 2$ A1

Exact particular solution:

$y = e^{\sin x + 2}$ (or $y = e^2 e^{\sin x}$) A1

Question 8

3 marks

Answer: $\lambda = -1 \pm 2i$; a stable spiral

$\lambda^2 + 2\lambda + 5 = 0$ M1

$\lambda = -1 \pm 2i$ A1

stable spiral: damped oscillation A1

Question 9

4 marks

Answer: $\lambda = -1, -2$; no oscillation

$\dot{x} = y, \dot{y} = -2x - 3y$ A1

$\lambda^2 + 3\lambda + 2 = 0$ M1

$\lambda = -1, \ -2$ A1

stable node: returns to rest without oscillating A1

Question 10

5 marks

Answer: $y = \dfrac{1}{3}e^x + \dfrac{2}{3}e^{-2x}$

integrating factor e^{2x} $M1$A1

$\dfrac{d}{dx}(ye^{2x}) = e^{3x}$ A1

$ye^{2x} = \dfrac{1}{3}e^{3x} + c, c = \dfrac{2}{3}$ A1

$y = \dfrac{1}{3}e^x + \dfrac{2}{3}e^{-2x}$ A1

Every solution is also on ibmathrevision.com/hlai/differential-equations-and-eulers-method-exam-practice
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