Proof: exam practice

10 questions · 44 marks · worked solutions and markschemes

Name _____ Date _____ Score / 44

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlaa/proof-exam-practice

1. **2 marks · Paper 1 style · no calculator**

Find a counter-example to show that the statement \"$n^2 + n + 41$ is prime for every $n \in \mathbb{N}$\" is false.

Give the counter-example. **[2]**

2. **2 marks · Paper 1 style · no calculator**

Show by counter-example that the statement \"if $a^2 > b^2$ then $a > b$\", for $a, b \in \mathbb{R}$, is false.

Give the counter-example. **[2]**

3. **4 marks · Paper 1 style · no calculator**

Prove that the sum of the squares of any two consecutive odd integers is 2 more than a multiple of 8.

Write the proof. **[4]**

4. **4 marks · Paper 1 style · no calculator**

Prove by contradiction that if n^2 is even then n is even, for $n \in \mathbb{Z}$.

Write the proof. **[4]**

5. **5 marks · Paper 1 style · no calculator**

Prove by contradiction that $\sqrt{3}$ is irrational.

Write the proof. **[5]**

6. **6 marks · Paper 1 style · no calculator**

Prove by mathematical induction that $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{Z}^+$.

Write the proof. **[6]**

7. **6 marks · Paper 1 style · no calculator**

Prove by mathematical induction that $3^{2n}+7$ is divisible by 8 for all $n \in \mathbb{Z}^+$.

Write the proof. **[6]**

8. **7 marks · Paper 2 style · calculator**

Prove by mathematical induction that $\frac{d^n}{dx^n}\left(xe^{2x}\right) = 2^{n-1}(2x+n)e^{2x}$ for all $n \in \mathbb{Z}^+$.

Write the proof. **[7]**

9. **4 marks · Paper 1 style · no calculator**

Prove that $\frac{1}{\sqrt{n+1}+\sqrt{n}} = \sqrt{n+1}-\sqrt{n}$ for $n \geq 0$, and hence find the exact value of $\sum_{n=0}^{48} \frac{1}{\sqrt{n+1}+\sqrt{n}}$.

(a) Prove the identity. **[2]**

(b) Find the sum. **[2]**

10. **4 marks · Paper 1 style · no calculator**

Prove that for all real x and y, $x^2 + y^2 \geq 2xy$, and hence show that $a + \frac{1}{a} \geq 2$ for all $a > 0$.

(a) Prove $x^2 + y^2 \geq 2xy$. **[2]**

(b) Show $a + \frac{1}{a} \geq 2$. **[2]**

Answers and worked solutions

Question 1 — 2 marks

Answer: $n = 40$ gives $1681 = 41^2$

$n = 40$: $40^2 + 40 + 41 = 1681$ **M1**

$= 41^2$, not prime **A1**

Question 2 — 2 marks

Answer: e.g. $a = -3, b = 1$

e.g. $a = -3, b = 1$ **A1**

$9 > 1$ but $-3 < 1$ **R1**

Question 3 — 4 marks

Answer: a proof: the worked solution above is the full answer

$(2n-1)^2 + (2n+1)^2$ **M1**

$= 8n^2 + 2$ **A1**

$8n^2$ is a multiple of 8 **R1**

so the sum is 2 more than a multiple of 8 **A1 AG**

Question 4 — 4 marks

Answer: a proof: the worked solution above is the full answer

assume n^2 is even and n is odd, $n = 2m + 1$ **M1**

$n^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$ **A1**

so n^2 is odd **A1**

contradiction, so n is even **R1**

Question 5 — 5 marks

Answer: a proof: the worked solution above is the full answer

assume $\sqrt{3} = \frac{p}{q}$, $p, q \in \mathbb{Z}$, no common factor **M1**

$p^2 = 3q^2$ so $3 \mid p^2$, hence $3 \mid p$ **A1**

$p = 3m$: $9m^2 = 3q^2$, $q^2 = 3m^2$ **A1**

so $3 \mid q$ **A1**

p and q share the factor 3: contradiction, so $\sqrt{3}$ is irrational **R1**

Question 6 — 6 marks

Answer: a proof: the worked solution above is the full answer

$n = 1$: LHS $= 2$, RHS $= \frac{1 \cdot 2 \cdot 3}{3} = 2$ **A1**

assume true for $n = k$ **M1**

$\frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$ **M1**

$= \frac{(k+1)(k+2)(k+3)}{3}$ **A1A1**

true for $n = 1$; true for $n = k$ implies true for $n = k + 1$; so true for all $n \in \mathbb{Z}^+$ **R1**

Question 7 — 6 marks

Answer: a proof: the worked solution above is the full answer

$n = 1$: $9 + 7 = 16 = 8 \times 2$ **A1**

assume $3^{2k} + 7 = 8m$ **M1**

$3^{2k+2} + 7 = 9 \cdot 3^{2k} + 7 = 9(8m - 7) + 7$ *M1***A1**

$= 72m - 56 = 8(9m - 7)$ **A1**

true for $n = 1$; $k \Rightarrow k + 1$; so true for all $n \in \mathbb{Z}^+$ **R1**

Question 8 — 7 marks

Answer: a proof: the worked solution above is the full answer

$n = 1$: $\frac{d}{dx}(xe^{2x}) = (1 + 2x)e^{2x} = 2^0(2x + 1)e^{2x}$ **A1**

assume true for $n = k$ **M1**

$\frac{d}{dx}\left[2^{k-1}(2x + k)e^{2x}\right]$ **M1**

$= 2^{k-1}\left[2e^{2x} + 2(2x + k)e^{2x}\right]$ **A1**

$= 2^k(2x + k + 1)e^{2x}$ **A1A1**

true for $n = 1$; $k \Rightarrow k + 1$; so true for all $n \in \mathbb{Z}^+$ **R1**

Question 9 — 4 marks

Answer: (b) 7

(a)

multiply by $\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}}$ **M1**

denominator $(n+1) - n = 1$ **A1**

AG

(b)

terms cancel: $\sqrt{49} - \sqrt{0}$ **M1**

7 **A1**

Question 10 — 4 marks

Answer: a proof: the worked solution above is the full answer

(a)

$x^2 + y^2 - 2xy = (x - y)^2$ **M1**

≥ 0 **A1 AG**

(b)

let $x = \sqrt{a}$, $y = \frac{1}{\sqrt{a}}$ **M1**

$a + \frac{1}{a} \geq 2\sqrt{a} \cdot \frac{1}{\sqrt{a}} = 2$ **A1 AG**

Every solution is also on ibmathrevision.com/hlaa/proof-exam-practice
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