First Principles: exam practice

10 questions · 44 marks · worked solutions and markschemes

Name _____ Date _____ Score / 44

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlaa/first-principles-exam-practice

1. **3 marks · Paper 1 style · no calculator**

Evaluate the following limit, showing your working clearly:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

2. **3 marks · Paper 1 style · no calculator**

Using L'Hôpital's rule, evaluate the limit:

$$\lim_{x \to 0} \frac{\sin(4x)}{x}$$

3. **4 marks · Paper 2 style · calculator**

Find the second derivative, $f''(x)$, of the function $f(x) = x^4 \ln x$ for $x > 0$.

4. **3 marks · Paper 1 style · no calculator**

Use l'Hôpital's rule to find $\lim_{x \to 0} \frac{e^{3x} - 1}{\sin x}$.

Find the limit. **[3]**

5. **4 marks · Paper 1 style · no calculator**

Use differentiation from first principles to find $f'(x)$ for $f(x) = 3x^2 - x$.

Find $f'(x)$ from first principles. **[4]**

6. **5 marks · Paper 2 style · calculator**

A function $f(x)$ is defined as:

$$f(x) = \frac{\ln x}{x^2 - 1}$$

The function is undefined at $x = 1$. Determine the value that $f(1)$ must be assigned in order to make the function continuous at $x = 1$.

7. **5 marks · Paper 1 style · no calculator**

Evaluate the following limit, which initially presents the indeterminate form $0 \cdot \infty$:

$$\lim_{x \to 0^+} x^2 \ln x$$

(Hint: Rewrite the expression as a fraction before applying L'Hôpital's rule).

8. **4 marks · Paper 1 style · no calculator**

Find $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$.

Find the limit. **[4]**

9. **6 marks · Paper 1 style · no calculator**

Evaluate the following limit requiring repeated applications of L'Hôpital's rule:

$$\lim_{x \to 0} \frac{x - \tan x}{x^3}$$

10. **7 marks · Paper 2 style · calculator**

The Maclaurin series expansion for $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

Evaluate the following limit:

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$$

Answers and worked solutions

Question 1 — 3 marks

Answer: 6

Factorise the numerator:
$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3}$ **M1**
Cancel $x-3$ (valid since $x \neq 3$):
$\lim_{x\to 3}(x+3)$ **A1**
$= 6$ **A1**

Question 2 — 3 marks

Answer: 4

Substituting $x = 0$ gives $\frac{0}{0}$, so L'Hôpital's rule applies.
Differentiate numerator and denominator:
$\lim_{x\to 0} \frac{4\cos(4x)}{1}$ *M1***A1**
$= 4$ **A1**

Question 3 — 4 marks

Answer:
$f''(x) = 12x^2\ln(x) + 7x^2$

Use the product rule with $u = x^4$, $v = \ln x$:
$f'(x) = 4x^3\ln x + x^4 \cdot \frac{1}{x}$ **M1**
$f'(x) = 4x^3\ln x + x^3$ **A1**
Differentiate again (product rule on $4x^3\ln x$):
$f''(x) = 12x^2\ln x + 4x^3\cdot\frac{1}{x} + 3x^2$ **M1**
$f''(x) = 12x^2\ln x + 7x^2$ **A1**

Question 4 — 3 marks

Answer: 3

form $\frac{0}{0}$ **R1**
$\lim_{x\to 0}\frac{3e^{3x}}{\cos x}$ **M1**
3 **A1**

Question 5 — 4 marks

Answer: $6x-1$

$\lim_{h\to 0}\frac{3(x+h)^2-(x+h)-3x^2+x}{h}$
M1
$= \lim_{h\to 0}\frac{6xh+3h^2-h}{h}$ **A1**
$= \lim_{h\to 0}(6x+3h-1)$ **A1**
$6x-1$ **A1**

Question 6 — 5 marks

Answer: $1/2$

For continuity, $f(1)$ must equal $\lim_{x\to 1}\frac{\ln x}{x^2-1}$.
At $x=1$ the limit has the indeterminate form $\frac{0}{0}$, so L'Hôpital's rule applies. **R1**
Differentiate numerator and denominator:
$\lim_{x\to 1}\frac{\frac{1}{x}}{2x}$ *M1***A1**
$= \frac{1}{2}$ **A1**
Hence $f(1) = \frac{1}{2}$ **A1**

Question 7 — 5 marks

Answer: 0

Rewrite as a fraction:
$x^2\ln x = \frac{\ln x}{x^{-2}}$, which has the form $\frac{-\infty}{\infty}$ as $x\to 0^+$ **M1**
Apply L'Hôpital's rule:
$\lim_{x\to 0^+}\frac{\frac{1}{x}}{-2x^{-3}}$ *M1***A1**
Simplify:
$= \lim_{x\to 0^+}\left(-\frac{x^2}{2}\right)$ **A1**
$= 0$ **A1**

Question 8 — 4 marks

Answer: 2

form $\frac{0}{0}$; l'Hôpital **M1**
$\lim\frac{2\sin 2x}{2x}$, still $\frac{0}{0}$ **A1**
$\lim\frac{4\cos 2x}{2}$ **M1**
2 **A1**

Question 9 — 6 marks

Answer: $-1/3$

Substituting $x=0$ gives $\frac{0}{0}$, so apply L'Hôpital's rule: **M1**
$\lim_{x\to 0}\frac{1-\sec^2 x}{3x^2}$ **A1**
Still $\frac{0}{0}$ (since $\sec 0 = 1$), so apply L'Hôpital's rule again: **M1**
$\lim_{x\to 0}\frac{-2\sec^2 x\tan x}{6x}$ **A1**
Still $\frac{0}{0}$; apply the rule a third time, using the product rule on the numerator:
$\lim_{x\to 0}\frac{-4\sec^2 x\tan^2 x - 2\sec^4 x}{6}$ **A1**
Substitute $x=0$ ($\tan 0 = 0$, $\sec 0 = 1$): $\frac{0-2}{6} = -\frac{1}{3}$ **A1**

Question 10 — 7 marks

Answer: $1/24$

Replace $\cos x$ by its Maclaurin series in the numerator: **M1**
$\cos x - 1 + \frac{x^2}{2} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots\right) - 1 + \frac{x^2}{2}$
The constant terms and the x^2 terms cancel: **A1**
$\cos x - 1 + \frac{x^2}{2} = \frac{x^4}{24} - \frac{x^6}{720} + \ldots$ **A1**
Divide each term by x^4: **M1**
$\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{1}{24} - \frac{x^2}{720} + \ldots$ **A1**
As $x\to 0$, every term containing a power of x tends to 0: **R1**
$\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{1}{24}$ **A1**

Every solution is also on ibmathrevision.com/hlaa/first-principles-exam-practice
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