IB MATHS AA HL · UNIT 5

Differential Equations and Maclaurin: exam practice

10 questions · 46 marks · worked solutions and markschemes

Name _____ Date _____ Score / 46

- Answer every question before you look at the solutions. They start after the last question, with the marks for each line: M for method, A for accuracy, R for reasoning.
- Online, with more practice: ibmathrevision.com/hlaa/differential-equations-and-maclaurin-exam-practice

1. **3 marks · Paper 1 style · no calculator**

Use Euler's method with step length 0.5 to estimate $y(1)$, given that $\frac{dy}{dx} = 1 - y$ and $y(0) = 0$.

2. **3 marks · Paper 1 style · no calculator**

Consider the linear differential equation:

$$\frac{dy}{dx} + \frac{2}{x}y = 5x$$

Find the integrating factor required to solve this differential equation.

3. **3 marks · Paper 1 style · no calculator**

Use the Maclaurin series for e^x and $\sin x$ to find the series for $e^x \sin x$ up to and including the term in x^3.

Find the series. **[3]**

4. **4 marks · Paper 2 style · calculator**

Use Euler's method with step $h = 0.1$ to estimate $y(0.3)$ for $\frac{dy}{dx} = x + y$, $y(0) = 1$.

Find the estimate. **[4]**

5. **5 marks · Paper 1 style · no calculator**

Use the integrating factor method to solve the first-order differential equation $\frac{dy}{dx} + \frac{2}{x}y = 4x$ for $x > 0$. Give the general solution for y in terms of x.

6. **5 marks · Paper 1 style · no calculator**

Solve $\frac{dy}{dx} + \frac{2y}{x} = x$, $x > 0$, given that $y = 1$ when $x = 1$.

Find y. **[5]**

7. **6 marks · Paper 2 style · calculator**

According to Newton's Law of Cooling, the rate at which an object cools is proportional to the difference between its temperature T and the ambient room temperature T_A. This is modelled by $\frac{dT}{dt} = -k(T - T_A)$.

A cup of coffee is initially $100°\text{C}$ in a room kept at a constant $20°\text{C}$. After 10 minutes, the coffee has cooled to $60°\text{C}$.

Calculate the exact temperature of the coffee after 20 minutes.

8. **6 marks · Paper 1 style · no calculator**

Find the Maclaurin series for $f(x) = e^x \ln(1 + x)$ up to and including the term in x^3.

9. **6 marks · Paper 1 style · no calculator**

Use the substitution $y = vx$ to solve $\frac{dy}{dx} = \frac{x + y}{x}$, $x > 0$, given that $y = 2$ when $x = 1$.

Find y in terms of x. **[6]**

10. **5 marks · Paper 1 style · no calculator**

Use $y = vx$ to show that the differential equation $\frac{dy}{dx} = \frac{y^2 + x^2}{xy}$ becomes $v\frac{dv}{dx} = \frac{1}{x}$, and hence find its general solution.

(a) Show the result. **[2]**

(b) Find the general solution. **[3]**

Answers and worked solutions

Question 1 — 3 marks

Answer: 0.75

$y(0.5) \approx 0 + 0.5(1-0) = 0.5$ $M1$A1

$y(1) \approx 0.5 + 0.5(1-0.5) = 0.75$ A1

Question 2 — 3 marks

Answer: x^2

The equation is in the form $\frac{dy}{dx} + P(x)y = Q(x)$ with $P(x) = \frac{2}{x}$.

Integrating factor:

$e^{\int \frac{2}{x}\,dx} = e^{2\ln x}$ $M1$A1

$= x^2$ A1

Question 3 — 3 marks

Answer: $x + x^2 + \frac{x^3}{3}$

$\left(1 + x + \frac{x^2}{2}\right)\left(x - \frac{x^3}{6}\right)$ M1

$x + x^2 + \frac{x^3}{2} - \frac{x^3}{6}$ A1

$x + x^2 + \frac{x^3}{3}$ A1

Question 4 — 4 marks

Answer: 1.362

$y_{n+1} = y_n + 0.1(x_n + y_n)$ M1

$y_1 = 1.1$ A1

$y_2 = 1.22$ A1

$y_3 = 1.362$ A1

Question 5 — 5 marks

Answer: $y = x^2 + C/x^2$

Integrating factor:

$I(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$ $M1$A1

Multiply the DE by x^2:

$x^2\frac{dy}{dx} + 2xy = 4x^3 \implies \frac{d}{dx}(x^2y) = 4x^3$ M1

Integrate both sides:

$x^2y = x^4 + C$ A1

$y = x^2 + \frac{C}{x^2}$ A1

Question 6 — 5 marks

Answer: $y = \frac{x^2}{4} + \frac{3}{4x^2}$

integrating factor $e^{\int \frac{2}{x}dx} = x^2$ $M1$ A1

$\frac{d}{dx}(x^2y) = x^3$ A1

$x^2y = \frac{x^4}{4} + c$, $c = \frac{3}{4}$ A1

$y = \frac{x^2}{4} + \frac{3}{4x^2}$ A1

Question 7 — 6 marks

Answer: 40

With $T_A = 20$, separate the variables:

$\int \frac{1}{T-20}\,dT = -\int k\,dt$ M1

$\ln(T-20) = -kt + c$, so $T = 20 + Ae^{-kt}$ A1

Use $T = 100$ at $t = 0$:

$A = 80$ A1

Use $T = 60$ at $t = 10$:

$40 = 80e^{-10k} \implies e^{-10k} = \frac{1}{2}$ $M1$A1

At $t = 20$,

$e^{-20k} = \left(e^{-10k}\right)^2 = \frac{1}{4}$:

$T = 20 + 80 \cdot \frac{1}{4} = 40°\text{C}$ A1

Question 8 — 6 marks

Answer: $x + x^2/2 + x^3/3$

Standard series:

$e^x = 1 + x + \frac{x^2}{2} + \ldots$ and $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ M1

Multiply, keeping terms up to x^3:

$1\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) + x\left(x - \frac{x^2}{2}\right) + \frac{x^2}{2}(x)$ M1

$= x - \frac{x^2}{2} + \frac{x^3}{3} + x^2 - \frac{x^3}{2} + \frac{x^3}{2}$ A1

Collect like terms:

$x + \left(-\frac{1}{2} + 1\right)x^2 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{2}\right)x^3$

$e^x\ln(1+x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$ A1 A1 A1

Question 9 — 6 marks

Answer: $y = x\ln x + 2x$

$v + x\frac{dv}{dx} = 1 + v$ $M1$A1

$\frac{dv}{dx} = \frac{1}{x}$ A1

$v = \ln x + c$ A1

$c = 2$ A1

$y = x\ln x + 2x$ A1

Question 10 — 5 marks

Answer: (b) $y^2 = 2x^2(\ln x + c)$

(a)

$v + x\frac{dv}{dx} = \frac{v^2+1}{v} = v + \frac{1}{v}$ M1

$x\frac{dv}{dx} = \frac{1}{v}$, so $v\frac{dv}{dx} = \frac{1}{x}$ A1 AG

(b)

$\frac{v^2}{2} = \ln x + c$ A1

$\frac{y^2}{2x^2} = \ln x + c$ M1

$y^2 = 2x^2(\ln x + c)$ A1

Every solution is also on ibmathrevision.com/hlaa/differential-equations-and-maclaurin-exam-practice
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